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THE CLASSIFICATION OF REAL PRIMITIVE INFINITE LIE ALGEBRAS

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The objects under consideration are the abstract transitive infinite Lie algebras (TILA) denoted (L, L^0) which are introduced in [4]. If we let $V = L/L^0$, then the realization theorem of [4] proves that (L, L^0) is topologically isomorphic to a subalgebra of $D(V)$, the continuous derivations of the formal power series $F(V^*)$. The complex primitive TILA have been classified.

We begin by recalling this classification. The results of [3] show that $g_L^0 = L^0/L^1$ contains elements of rank one. This fact is used to show that g_L^0 is one of the following: 1. $\mathrm{sl}(n, C)$, 2. $\mathrm{gl}(n, C)$, 3. $\mathrm{sp}(n, C)$, 4. $\mathrm{Csp}(n, C)$ $(\mathrm{sp}(n, C) + \{I\})$, or that there is a formal 1-form of maximal rank such that the principle module generated by it over $F(V^*)$ is preserved under Lie derivation by elements of L.

In [6] these possibilities are analyzed and the following results established:

1. If the linear isotropy algebra $(= g_L^0)$ is $\mathrm{sl}(n, C)$, then L is the algebra of all vector fields with divergence zero.

2. If the linear isotroply algebra is $\mathrm{gl}(n, C)$, then L is either the algebra of vector fields of constant divergence or the algebra of all vector fields.

3. If the linear isotropy algebra is $\mathrm{sp}(n, C)$, then L is the algebra of all Hamiltonian vector fields.

4. In the last case L is the contact algebra.

We plan to classify the real primitive TILA using these results. We begin with a theorem of Guillemin proved in [1].

Theorem 1. *Any primitive* TILA (L, L^0) *contains a closed ideal I of finite codimension such that* $(I, I \cap L^0)$ *is a primitive, simple* TILA.

If (L, L^0) is a TILA, we define a filtered derivation to be a derivation d for which there exists an integer $-1 \leq i < \infty$ such that $d\colon L^k \to L^{k+i}$ for all k.

Corollary. *Any primitive* TILA *is contained in the algebra of filtered derivations of a primitive simple* TILA, *and contains the algebra of inner derivations as a closed ideal of finite codimension.*

Note that the filtered derivations of a TILA (M, M^0) form a filtered algebra Δ. Set $g_\Delta^i = \Delta^i/\Delta^{i+1}$. $g_\Delta^0 \circledS g_\Delta^{-1}$ is mapped injectively into the derivations of $g_M^0 \circledS g_M^{-1}$ which preserve g_M^{-1} which we shall denote by $\mathrm{Der}_*(g_M^0 \circledS g_M^{-1})$.

The program for completing the classification of real primitive TILA is to determine

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1. all real simple TILA (M, M^0),
2. $\mathrm{Der}_*(g^0_M \circledS g^{-1}_M)$ for these M,
3. the possible primitive TILA (L, L^0) such that (a) $g^0_L \circledS g^{-1}_L$ is isomorphic to a subalgebra of $\mathrm{Der}_*(g^0_M \circledS g^{-1}_M)$ containing $g^0_M \circledS g^{-1}_M$, and (b) L contains a subalgebra N isomorphic to M such that $g^0_M \circledS g^{-1}_N$ is mapped isomorphically onto $g^0_M \circledS g^{-1}_M$ under the isomorphism in (a).

I. Determination of the real simple TILA

Let (M, M^0) be a real simple TILA, and let $L = M \otimes C$. If L is not simple, let I be a proper closed ideal. Then define π and ρ as follows: $\pi(X+iY)=X$, $\rho(X+iY)=Y$. Let $J=\pi[I] \subset M$. If we can show that π is $1-1$, then by the theorem in [1] stating that any continuous $1-1$ map between TILA is uniformly continuous we can conclude that J is closed. Hence $J=M$. $\rho \circ \pi^{-1}$ defines a complex structure on M, and M is a complex simple primitive TILA. It is the algebra of vector fileds corresponding to one of the groups in [2].

π is $1-1$. If not, then there exist purely imaginary elements of I and hence pure real elements. $I \cap M \neq \{0\}$, so that $I \cap M$ and $I \supset M$. This implies $I=L$ contradicting all propriety.

Thus we have reduced the classification problem of part I to the case where M is a real form of a complex simple TILA. In this case is (L, L^0) primitive? We shall show that it is. For this we use an alternate characterization of primitivity proved in [7].

Theorem 2. *A TILA (L, L^0) is primitive if and only if $F(L/L^0)$ has no invariant proper subalgebras other than $\{0\}$ and constants.*

Theorem 3. *Suppose (M, M^0) is a real simple primitive* TILA, *$L=M\otimes C$, and $L^0 = M^0 \otimes C$. If (L, L^0) is simple, then it is also primitive.*

Proof. Let $F(M/M^0)=G$; then $F(L/L^0)=G\otimes C$. Suppose (L, L^0) is not primitive; then there exists an invariant nontrivial subalgebra F^1 of $G\otimes C$. Let F_2 consist of the complex conjugates of the elements of F_1. Then $F_0 = F_1 \cap F_2$ is of the form $G_0 \otimes C$, and $F_1 \cdot F_2$ is also of the form $G_1 \otimes C$. $G_0 \neq G$, so that $G_0 = \{0\}$ or R. $G_1 \neq \{0\}, R$, so that $G_1 = G$. We conclude that $F_1 \supset C$ and $F_1 \cdot F_2 = G$. Suppose $f \in F^0(L/L^0)$. Then $f=\sum f_i \bar{g}_i$ where $f_i, g_i \in F_1$. If $f \in F^0(L/L^0)$ (power series with no constant term), then $f = \Sigma\{[f_i - f_i(0)]\bar{g}_i + f_i(0)[g_i - g_i(0)]\}$. Since $F_1 \supset C$, there exists a $h_i \in F_1 \cap F^0$ such that $g_i = \lambda + h_i$. Thus $F_1 \cap F^0/F_1 \cap F^1$ splits F^0/F^1. If $x^1, \cdots, x^n$ are the cosets spanning $F_1 \cap F^0/F_1 \cap F^1$, then $F_1 = F(x^1, \cdots, x^n)$, showing that $\{X \mid X \in L, X \cdot F_1 = 0\}$ is a nontrivial proper closed ideal of L contradicting simplicity.

(L, L^0) is a complex primitive TILA, and M is a real simple subalgebra which is primitive with respect to the filtration $M^i = L^i \cap M$. If $L/L^0 = V$, then L is isomorphic to one of the subalgebras of $D(V)$ mentioned in the introduction. Let $M/M^0 = W$. Then W is a real form of V, and $M^0/M^1 = g^0_M$ is is a real form of g^0_L leaving W fixed. If g^0_L is $\mathrm{gl}(V)$ of $\mathrm{sl}(V)$, then it is immediate that $g_M^0 = \mathrm{gl}(W)$ or $\mathrm{sl}(W)$. If g_L^0 is $\mathrm{sp}(V)$ where the alternating bilinear form defining $\mathrm{sp}(V)$ is J, then we reason as follows. Choose a basis for V in W. With respect to this basis g_M^0 is a Lie algebra of real matrices which is a real form for g_L^0. Let $A_1, \cdots, A_r$ be a basis for g_M^0, and J also denote the matrix defining the form J with respect to our basis. Consider the equations for a matrix B given by ${}^tA_kB + BA_k = 0$, $\mathrm{k} = 1, \cdots, r$, let $B_1, \cdots, B_l$ be the real matrices spanning the solution space, and define $f(x^1, \cdots, x^l) = \det(x^1B_1 + \cdots + x^lB_l)$. Then $J = \lambda^1 B_1 + \cdots + \lambda^l B_l$ for some $(\lambda^1, \cdots, \lambda^l)$, and thus $f(\lambda^1, \cdots, \lambda^l) \neq 0$ and there exist real numbers $a^1, \cdots, a^l$ such that $f(a^1, \cdots a^l) \neq 0$. Set $\mathrm{J}^* = A^1B_1 + \cdots + A^lB_l$. By a dimension argument due to Matsushima we see $g_M^0 = \mathrm{sp}(W, J^*)$.

In the three case where

$$g_L^0 = \begin{cases} 1. & \mathrm{gl}(V), \\ 2. & \mathrm{sl}(V), \\ 3. & \mathrm{sp}(V, J), \end{cases} \qquad \text{we find} \qquad g_M^0 = \begin{cases} 1. & \mathrm{gl}(W), \\ 2. & \mathrm{sl}(W), \\ 3. & \mathrm{sp}(W, J^*). \end{cases}$$

The reasoning in [6] shows that the only simple algebras with these linear isotropy algebras are

1. $D(W)$,
2. all vector fields in $D(W)$ with 0 divergence,
3. all Hamiltonian vector fields.

To complete the first step of the classification we must study the real forms of the contact algebra.

Let V be an odd dimensional vector space, and $x^1, \cdots, x^n, y^1, \cdots y^n, z$ a basis for the dual V^*. The contact algebra is the subalgebra of $D(V)$ consisting of those vector fields X such that $D_X\omega = f_X\omega$ where D_X is the Lie derivative with respect to X, ω is the 1-form $dz + \Sigma(y^i dx^i - x^i dy^i)$, and f_X is a function on V.

In [6] the structure of this algebra is analyzed. Let g be the corresponding graded algebra. Then $V = g^{-1}$ is spanned by the cosets of $\partial/\partial z$, $\partial/\partial x^i + y^i\partial/\partial z$, $\partial/\partial y^i - x^i\partial/\partial z$. g^0 is spanned by the cosets of the following vector fields:

$$x^i\partial/\partial x^j - y^j\partial/\partial y^i, x^i\partial/\partial y^j + x^j\partial/\partial y^i, y^i\partial/\partial^j + y^j\partial/\partial x^i,$$

$$z\partial/\partial y^i - x^i\left(z\partial/\partial z + \sum_1^n x^j\partial/\partial x^j + y^j\partial/\partial y^j\right),$$

$$z\partial/\partial x^i + y^i\left(z\partial/\partial z + \sum_1^n x^j\partial/\partial x^j + y^j\partial/\partial y^j\right),$$

$$2z\partial/\partial z + \sum_1^n (x^i\partial/\partial x^i + y^i\partial/\partial y^i).$$

Let V_0 be the subspace of g^{-1} spanned by the cosets of

$$\partial/\partial x^i + y^i\partial/\partial z, \qquad \partial/\partial y^i - x^i\partial/\partial z.$$

Define $\gamma\colon V_0 \to g_L^0$,

$$\text{coset}\,(\partial/\partial x^i + y^i\partial/\partial z) \mapsto \text{coset}\,(z\partial/\partial x^i + y^i(z\partial/\partial z + \textstyle\sum (x^j\partial/\partial x^j + y^j\partial/\partial y^j)))\ ,$$
$$\text{coset}\,(\partial/\partial y^i - x^i\partial/\partial z) \mapsto \text{coset}\,(z\partial/\partial y^i - x^i(z\partial/\partial z + \textstyle\sum (x^j\partial/\partial x^j + y^j\partial/\partial y^j)))\ .$$

If $V_* = \gamma(V_0)$, then $g_L^0 \cong \mathrm{Csp}\,(V_*) \circledS V_*$, where $\mathrm{Csp}\,(V_*)$ is the symplectic algebra plus center. Henceforth let (L, L^0) denote the contact algebra, and (M, M^0) be a real form $(M, M^0) \otimes \boldsymbol{C} = (L, L^0)$. Then we have $g_M^0 \otimes \boldsymbol{C} = g_L^0$, $g_M^{-1} \otimes \boldsymbol{C} = g_L^{-1}$.

We will first determine the real forms of $\mathrm{Csp}\,(V_*) \circledS V_*$, which leave a real form of V invariant, and then consider the prolongations of these algebras to find which linear algebras are g^0 of a graded algebra which complexifies properly, and finally to find which filtered algebras corresponding to these graded algebras complexify properly.

Let G be a real form of $\mathrm{Csp}\,(V_*) \circledS V_*$ leaving some real form of V invariant, $i\colon G \to \mathrm{Csp} \circledS V_*$ the inclusion map, α the component with range in $\mathrm{Csp}\,(V_*)$, and β the component with range in V_*. Then

$$\alpha[X, Y] = [\alpha(X), \alpha(Y)]\ ,$$
$$\beta[X, Y] = [\alpha(X), \beta(Y)] - [\alpha(Y), \beta(X)]\ .$$

Kernel $\alpha = W$ is an ideal, and $\beta \restriction W$ is injective. Therefore W is abelian. $\beta(W)$ is a real form of V_* since $\beta(W) \cap i\beta(W) = \{0\}$ and $\beta(W) + i\beta(W) = V_*$. Both of these equalities hold because the subspaces on the right are complex subspace invariant under $\alpha(G) + i\alpha(G) = \mathrm{Csp}\,(V_*)$.

Let $\hat{\alpha}$ be the factorization of α through $\hat{G} = G/W$. Then $\hat{\alpha}(G)$ is a real form of $\mathrm{Csp}\,(V_*)$ leaving a real form of $\mathrm{Csp}\,(V_*)$ invariant, and $\hat{\alpha}[\hat{G}, \hat{G}]$ is a real form of $\mathrm{sp}\,(V_*)$ leaving a real form $\beta(W)$ of V_* invariant. We have seen that in this case $\hat{\alpha}[\hat{G}, \hat{G}] = \mathrm{sp}\,(\beta(W), J_*)$.

Since every element of $\hat{\alpha}(\hat{G})$ can be expressed as a matrix with real coefficients and the identity is in $\hat{\alpha}(\hat{G}) + i\hat{\alpha}(\hat{G})$, the identity must be in $\hat{\alpha}(\hat{G})$. Therefore $\hat{G} = [\hat{G}, \hat{G}] + \hat{\alpha}^{-1}(I)$, and $\hat{G}$ is a simple subalgebra of $\mathrm{End}\,(W)$ plus center. Thus all extensions of $\hat{G}$ are inessential. Since G is such an extension, $G = G_0 \circledS W$, where is G_0 mapped bijectively onto G/W under projection.

Choose a basis $\{w_1, w_{2n}\}$ for W such that J_* has the form $\sum_1^n w_i^* \wedge w_{i+n}^*$. There exists a $u \in g_M^{-1}$ such that $u = \alpha\partial/\partial z + \sum (u^i\partial/\partial x^i + \lambda^i\partial/\partial y^l)$ where $\alpha \neq 0$, Change basis in V so that

$$\partial/\partial\bar{z} = \alpha\partial/\partial z + \textstyle\sum (u^i\partial/\partial x^i + \lambda^i\partial/\partial y^i)\ ,$$
$$\partial/\partial\bar{x}^i = [\partial/\partial\bar{z}, w_i]\ ,$$
$$\partial/\partial\bar{y}^i = [\partial/\partial\bar{z}, w_{i+n}]\ .$$

With respect to this basis, $w_i = \bar{z}\partial/\partial x^i, w_{i+n} = z\partial/\partial\bar{y}^i$, and also G_0 is spanned by

$$\bar{x}^i\partial/\partial\bar{x}^j - \bar{y}^j\partial/\partial\bar{y}^i , \qquad \bar{x}^j\partial/\partial\bar{y}^i + \bar{x}^i\partial/\partial\bar{y}^j ,$$

$$\bar{y}^j\partial/\partial\bar{x}^i + \bar{y}^i\partial/\partial\bar{x}^j , \qquad \partial\bar{z}\partial/\partial\bar{z} + \sum_1^n (\bar{z}^i\partial/\partial\bar{x}^i + \bar{y}^i\partial/\partial\bar{y}^i) .$$

What is g_M, if g_M^{-1} and g_M^0 are as above?

Let $U_* = \operatorname{span}\{\partial/\partial\bar{x}^i, \partial/\partial\bar{y}^i\}_{i=1,\cdots,n}$. In [6], it is established that

$$(g_M^0)^{(k)} = \{z_k\} \oplus \bar{z}^k U_* \oplus \bar{z}^{k-1}\operatorname{sp}(U_*) \oplus \cdots \oplus \operatorname{sp}(U_*)^{(k)} ,$$

where

$$z_k = \frac{2\bar{z}^{k+1}}{(k+1)!}\partial/\partial\bar{z} + \frac{\bar{z}^k}{k!}\sum_1^n (\bar{x}^i\partial/\partial\bar{x}^i + \bar{y}^i\partial/\partial^i) .$$

Since g_L^k is the full prolongation which has the same representation as $(g_M)^{(k)}$ only with respect to the basis $\{\partial/\partial z, \partial/\partial x^i, \partial/\partial y^i\}$, we conclude that $(g_M^0)^{(x)} = g_M^k$. The graded algebra of M has the same structure as the graded algebra of the contact algebra.

We must now determine the possible filtered algebras having this graded structure. The solution depends on the cohomology groups studied in [6].

We first give two preliminary definitions, and then two relevant theorems. If $v_1, \cdots, v_l$ are in g^{-1}, we define $g^k_{\{v_1,\cdots,v_l\}} = \{X \mid X \in g^k, [X, v_i] = 0, i = 1, \cdots, l\}$. We also define $\delta_{v_i}(X) = [X, v_i]$.

Theorem 4. *If there exists a basis $v_1, \cdots, v_l$ for g_M^{-1} such that*

$$\delta_{v_1} g_M^{k+1} = g_M^k , \quad \delta_{v_2} g^{k+1}_{\{v_1\}} = g^k_{\{v_1\}}, \cdots , \quad \delta_{v_n} g^{k+1}_{\{v_1,\cdots,v_{n-1}\}} = g^k_{\{v_1,\cdots,v_{n-1}\}} ,$$

then $H^{k,l}(g_M) = 0, k \geq 0$.

Theorem 5. *Let $(N, N^0), (P, P^0)$ be* TILA, *and suppose that*

(a) $i_{k_0}: (N, N^0) \to (P, P^0)$,

(b) i_{k_0} *induces an isomorphism* $g_N^{-1} \to g_p^{-1}$,

(c) $[i_{k_0}(X), i_{k_0}(Y)] - i_{k_0}[X, Y] \in P^{k_0}$,

(d) $H^{k,1}(g_p) = H^{k,2}(g_p) = 0, k \leq k_0$.

Then there exists an algebra homomorphism $i:(N, N^0) \to (P, P^0)$ with $i = i_k$, $\operatorname{mod} p^k$.

In the case where P is the real contact algebra which we shall denote by Q, $\{[\partial/\partial z], [\partial/\partial x^i + y^i\partial/\partial z], [\partial/\partial y^i + x^i\partial/\partial z]\}$, where $[\cdot]$ denotes coset with respect to Q^0, is a basis satisfying the conditions of Theorem 4. If $P = g_Q$, the graded algebra of Q, then $\{\partial/\partial z, \partial/\partial x^i, \partial/\partial y^i\}$ form a basis satisfying the conditions of Theorem 4.

We will show that for any N such that $g_N \cong g_Q$, we can choose one of the above algebras Q or g_Q, and for $k = 0$ we get an i_k: $(N, N^0) \to (P, P^0)$ satisfying Theorem 5.

Let N_* be a complement to N^0 in N, assume $P = g_Q$, and let P_* be the complement $\{\partial/\partial z, \partial/\partial x^i, \partial/\partial y^i\}$ to P^0, and α be the isomorphism of g_N to g_Q. Then α_{-1}: $N/N^0 \to P/P^0$, α_0: $N^0N^1 \to P^0/P^1$. Define β: $N \to P$ as follows:

$$\beta \restriction N_* \text{ ts induced by } N_* \to N/N^0 \xrightarrow{\alpha_{-1}} P/P^0 \to P_* ,$$

$$\beta \restriction N^0 \text{ is any lifting of } \alpha_0 .$$

We have $[\beta(X), \beta(Y)] - \beta[X, Y] \in P^0$ for $X \in N^0$, $Y \in N$. Thus β defines a map γ: $N/N^0 \wedge N/N^0 \to P/P^0$ by $\gamma(u \wedge v) = [\beta(X_u), \beta(X_v)] - \beta[X_u, X_v]$, where $X_u, X_v \in N_*$, $X_u \bmod N^0 = u$, $X_v \bmod N^0 = v$.

If we can find a ξ: $N_* \to P^0$ such that

$$[\xi(X_u), \beta(X_v)] - [\xi(X_v), \beta(X_u) \qquad \bmod P^0 = \gamma(u \wedge v) ,$$

then set $\xi \restriction N^0 = 0$ and define $\beta' = \beta + \xi$. Hence

$$[\beta'(X_u), \beta'(X_v)] - \beta'[X_u, X_v] \quad \bmod P^0 = 0 , \qquad \beta' = i_0 ,$$

for application of Theorem 5.

The question then is: can such a ξ be found? Finding such a ξ amounts to to finding a coboundary for a cocycle in $g_Q^{-1} \otimes \Lambda^2(g_Q^{-1})$. The cocycle in question is $\gamma \circ \alpha_{-1}^{-1} = \omega_\beta$: $\Lambda^2(g_Q^{-1}) \to g_Q^{-1}$. In fact the cohomology class of ω_β is invariant under the action of g_Q^0. Hence what we want to show is that every invariant cohomology class in $H^{-1,2}(g_Q)$ is 0. This is not true, that is, we will not always be able to find a ξ with the desired property. However, if β is such that ω_β does not bound, then ω_β differs from some real multiple of $\partial/\partial z \otimes \sum_1^n (dy^i \wedge dx^i)$ by a boundary. In other words, there are exactly two invariant cohomology classes in $H^{-1,2}(g_Q)$, namely, $[0]$ and $[\partial/\partial z \otimes \sum_1^n dy^i \wedge dx^i]$.

When ω_β does not bound, we can find a ξ: $N_* \to P^0$ such that if $\beta' = \beta + \xi$, then $\omega_{\beta'} = \lambda \partial/\partial z \otimes \sum (dy^i \wedge dx^i)$. Let $\partial/\partial\tilde{x}^i = \beta'^{-1}(\partial/\partial x^i)$, $\partial/\partial\tilde{y}^i = \beta'^{-1}(\partial/\partial y^i)$, $\partial/\partial\tilde{z} = \beta'^{-1}(\partial/\partial z)$ Span $\{\partial/\partial\tilde{x}^i, \partial/\partial\tilde{y}^i, \partial/\partial\tilde{z}^i\}$ and form a compact to N^0. Then the basis vectors have the same bracket relations as $\partial/\partial x^i + y^i\partial/\partial z$, $\partial/\partial y^i - x^i\partial/\partial z$, $\partial/\partial z$. In this case we can find a homomorphism i_0: $(N, N^0) \to (Q, Q^0)$ such that the conditions of Theorem 5 are satisfied. Thus, if $g_M = g_Q$, then M is either flat or the real contact algebra. Since $M \otimes C$ is not flat, M must be the real contact algebra, i.e., real vector fields preserving the form

$$\omega = d\tilde{z} - \sum_i^n (\tilde{y}^i d\tilde{x}^i - \tilde{x}^i d\tilde{y}^i)$$

up to a function multiple.

We now know the following possible real simple TILA in $D(V)$, V being a real vector space:

1. Complex vector fields with respect to some complex structure on V.
2. Complex vector fields of divergence 0.
3. Complex Hamiltonian vector fields.
4. Complex contact vector fields.
5. Real vector fields $= D(V)$.
6. Real vector fields of zero divergence.
7. Real Hamiltonian vector fields.
8. Real contact vector fields.

II. $\mathrm{Der}_*(g_M^0 \circledS g_M^{-1})$ for these TILA (M, M^0)

Now we want to determine the structure of $\mathrm{Der}_*(g_M^{-1} \circledS g_M^0)$ for M among the preceding algebras.

Consider the case when g_M^0 is semisimple or semisimple plus center. Then $d \in \mathrm{Der}_*(g_M^0 \circledS g_M^{-1})$ can be represented in matrix form:

$$d = \begin{pmatrix} d_1^1 & d_2^1 \\ d_1^2 & d_2^2 \end{pmatrix},$$

$$d_1^1 \colon g_M^0 \to g_M^0,\ d_1^2 \colon g_M^0 \to g_M^{-1},\ \text{etc.},\ d_2^1 = 0\ ,$$

since g_M^{-1} is preserved. Now $g_M^0 = \mathrm{Der}(g_M^0)$ if g_M^0 is semisimple, and $g_M^0 \overset{\phi}{\rightleftarrows} \mathrm{Der}(g_M^0)$ if g_M^0 is semisimple plus center, where ϕ is described as follows:

$$\phi(B + \lambda I)(A + \mu I) = [A, B] + \lambda\mu I\ ,$$

where $A, B \in [g_M^0, g_M^0]$, and λ, μ are linear maps of center into itself. Hence $d_1^1(A + \mu I) = [A, B_d] + \lambda_d \mu I$.

We will prove $\lambda_d = 0$ when g_M^0 is semisimple plus center. Note that $d_2^2[A, v] = [d_1^1 A, v] + [A, d_2^2 v]$. Substituting $A = I$ we find $d_2^2 v = \lambda_d v + d_2^2 v$, and hence $\lambda_d = 0$. If $C = d_2^2 \in \mathrm{End}(g_M^{-1})$, then $d_2^2[A, v] = [d_1^1 A, v] + [A, d_2^2 v]$ show that $[[A, v], C] = [[A, B_d], v] + [A, [v, C]]$, which is the same as $[[A, B_d], v] - [A, [C, v]] + [C, [A, v]] = 0$, or $[[A, B_d - C], v] = 0$ for all v. Thus $C - B_d = \lambda_d I$, where λ_d is a linear map of the center into itself, and hence $d_2^2 v = [v, \lambda_d I + B_d]$. Since $d_1^2[A, B] = [d_1^2 A, B] + [A, d_1^2 B]$ is a cocycle condition, $d_1^2 A = [A, v_d]$. Hence we conclude

$$b(A + v) = [A + v, B_d + \lambda_d I + v_d]\ ,$$

and

$$\mathrm{Der}_*(g_M^0 \circledS g_M^{-1}) = \begin{cases} g_M^0 \circledS g_M^{-1}, & \text{if } g_M^0 \text{ is semisimple plus center}\ , \\ g_M^0 + (I) \circledS g_M^{-1}, & \text{if } g_M^0 \text{ is semisimple}\ , \end{cases}$$

A similar argument show that $\text{Der}_*(g_M^0 \circledS g_M^{-1}) = g_M^0 + (I) \circledS g_M^{-1}$, when $g_M^0 = \text{Csp}(U_*) \circledS U_*$.

III. The possible primitive TILA (L, L^0), whose first two graded components are the appropriate spaces of derivations of the first two graded components of the (M, M^0) above

Now we must determine the TILA (L, L^0) such that $g_L^0 \circledS g_L^{-1} \subset \text{Der}_*(g_M^0 \circledS g_M^{-1})$ and such that (M, M^0) is an ideal in (L, L^0). We consider each (M, M^0) enumerated in § I separately.

1. M is the complex algebra $D(V)$, where V is a complex vector space $\text{Der}_*(g_M^0 \circledS g_M^{-1}) = g_M^0 \circledS g_M^{-1}, g_L^0 = g_M^0, g_L^{-1} = g_M^{-1}$, and g_L^1 is contained in the space of real prolongations of g_L^0. Are there any elements T of g_L^1 such that there exists a $v \in V = g_M^{-1}$ for which $T(iv) \neq iT(v)$? We know that $T(iv)u = T(u)iv = iT(u)v$, because $T(u) \in g_L^0$ which consists of complex linear maps. Also $T(u)v = T(v)u$, so that $T(iv)u = iT(v)u$ for all $u, v \in V$. Thus g_L^1 consists entirely of complex prolongations, and since g_M^1 is the full complex prolongation space, $g_L^1 = g_M^1$. Repeating this reasoning we find $g_L^i = g_M^i$, and conclude $L = M$.

2. $\text{Der}_*(g_M^0 \circledS g_M^{-1}) = g_M^0 + \text{center} \circledS g_M^{-1}$. In this case, $g_L^0 \subseteq g_M^0 + (I)$. But g_M is an ideal in g_L, so that g_L^1 must bracket g_M^{-1} into g_M^0. This shows that g_L^1 is included in the space of real prolongations of g_M^0. The argument in case 1 shows that g_L^1 consists only of complex prolongations. Thus we conclude $g_L \subseteq g_M + (I)$. Since M is flat, I is a derivation, and hence $M \subseteq L \subseteq M + (I)$.

3. By the same reasoning as above, $M \subseteq L \subseteq M + (I)$.

4. In this case, $L = M$ since $L \subseteq M + (I)$ and I is not a derivation of M.

5. As in case 1, $L = M$.

6, 7. As in case 2 and 3, $M \subseteq L \subseteq M + (I)$.

8. As in case 4, $L = M$.

One easily verifies that all of the above algebras are primitive, and the complete list of real primitive TILA follows:

1. $D(V)$, V being a complex vector space.
2. Algebra of complex vector fields of divergence zero.
3. Algebra of complex vector fields with divergence on some real line in $\boldsymbol{C}$.
4. Algebra of vector fields of complex constant divergence.
5. Algebra of complex Hamiltonian vector fields.
6. Algebra of vector fields preserving a Hamiltonian form up to constant lying on a real line in $\boldsymbol{C}$.
7. Algebra of vector fields preserving a Hamiltonian form up to a complex constant.
8. Complex contact algebra.
9. $D(v)$, V being a real vector space.
10. Algebra of real vector fields of divergence 0.

11. Algebra of real vector fields of constant divergence.

12. Algebra of real Hamiltonian vector fields.

13. Algebra of real vector fields preserving a Hamilonian form up to a constant multiple.

14. Real contact algebra.

References

[1] V. Guillemin, *A Jordan Hölder theorem for certain infinite Lie algebras*, unpublished.

[2] V. Guillemin, D. Quillen & S. Sternberg, *The classification of the complex primitive infinite pseudogroups*, Proc. Nat. Acad. Sci. U.S.A. **55** (1966) 687–690.

[3] ——, *The classification of the irreducible complex algebras of infinite type*, J. Analyse Math. **18** (1967) 107–112.

[4] V. Guillemin & S. Sternberg, *An algebraic model of transitive differential geometry*, Bull. Amer. Math. Soc. **70** (1964) 16–47.

[5] I. M. Singer & S. Sternberg, *On the infinite groups of Lie and Cartan, Part I (The transitive groups)*, J. Analyse Math. **15** (1965) 1–114.

[6] S. Sternberg, *Notes on transitive infinite Lie algebras*, unpublished lecture notes.

[7] ——, *Odd results on infinite algebras*, unpublished.

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